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GENERALIZED HOLSTEIN-PRIMAKOFF SQUEEZED STATES FOR SU(n)

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Abstract

We show how to define multi-photon, many-mode squeezed states for SU(n), using a generalized Holstein-Primakoff realization. We prove that for the class of realizations given, the resulting squeezing reduces to that of SU(2), and exemplify with a specific calculation for SU(3).

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Coherent[1] photon states $|\alpha\rangle = \exp(\alpha a^\dagger)|0\rangle$ are eigenstates of the photon annihilation operator a; they have the property that the uncertainty product $\Delta x \, \Delta p$ attains its minimum value $1/2(\hbar=1)$, while $[\Delta x]^2 = 1/2, [\Delta p]^2 = 1/2$ in such states. Squeezed states allow one of the uncertainties to go below $1/\sqrt{2}$, at the expense of the other, and may prove useful in low-noise detection experiments; they have the form[2], for example, of $\exp\{z(a^\dagger)^2\}|0\rangle$. The latter are associated with the group SU(1,1), generated by $\{(a^\dagger)^2, a^2\}$. These simple one-photon coherent states may be generalized[3] by considering states of the form $\exp\{\zeta A^\dagger_{(k)}\}|0\rangle$, where $A_{(k)}$ is a k-photon generalized boson operator[4], defined by $A^\dagger_{(k)} = ([\![\hat{n}/k]\!]\,(\hat{n}-k)!/\hat{n}!)^{\frac{1}{2}}\,(a^\dagger)^k$ and satisfying $[A_{(k)}, A^\dagger_{(k)}] = I$; and the squeezed states similarly generalized by considering Holstein-Primakoff[5] representations of a desired Lie group, expressed in terms of the $A_{(k)}, A^\dagger_{(k)}$.

This has been done for the groups SU(2) and SU(1,1)[6], where it was shown that improved squeezing could be obtained. This prompts the question as to whether further improvement could be obtained by considering Holstein-Primakoff realisations of Lie groups of higher rank, such as SU(n). We report here that naïve generalizations, such as those previously given for SU(n)[7] and SU(3)[8], do not, in fact, lead to other than the SU(2)-type results for optimal squeezing. Consider, for example, the form given by Okubo[7] for SU(n): the infinitesimal generators x^μ_ν of U(n) ($\mu, \nu = 1,2,\dots,n$) satisfy

$$[x^\mu_\nu, x^\alpha_\beta] = \delta^\mu_\beta x^\alpha_\nu - \delta^\alpha_\nu x^\mu_\beta \,. \tag{1}$$

Then

$$\{x^\mu_\nu = a^\dagger_\nu a_\mu \,|\, \mu \neq n, \nu \neq n \quad ; \quad x^n_\nu = a^\dagger_\nu [2\sigma - (\hat{n}-1)]^{\frac{1}{2}} = (x^\nu_n)^\dagger \,|\, \nu \neq n\} \tag{2}$$

where $\hat{n}-1 = \sum_{\nu=1}^{n-1} a^\dagger_\nu a_\nu$, gives a Holstein-Primakoff realization of SU(n), generalizing that previously used for SU(2).

We may now define squeezed states for SU(n) as coherent states using the Holstein-Primakoff realisation above (with the one-photon mode a replaced by the k-photon mode $A_{(k)}$ if desired); thus

$$|\alpha\rangle = \left\{ \prod_{\substack{\nu \\ \nu\neq n}} \exp(\alpha_\nu x_\nu^n) \prod_{\substack{\mu \\ \mu\neq n}} \exp(\alpha'_\mu x_n^\mu) \prod_{\substack{\mu,\nu \\ \mu,\nu\neq n}} \exp(\alpha_\nu^\mu x_\nu^\mu) \right\} |0\rangle$$

(up to suitable normalisation). We have chosen a canonical ordering in which the first two products, over α'_μ and α_ν^μ, leave the vacuum invariant, so that $|\alpha\rangle = \prod_\nu \exp(\alpha_\nu x_\nu^n)|0\rangle$

Each of the terms in this product now commutes with the others, so that $|\alpha\rangle$ is effectively a direct product of (n-1) independent SU(2) coherent (or k-photon squeezed) states, each associated with one of the (n-1) modes.

We exemplify this theoretical result with a specific computation based on a modification of the Holstein-Primakoff realisation of SU(3) given by Wagner[8]: we have two photon modes a_1, a_2 with

$$\tau_+ = a_1^\dagger \sqrt{N-(a_1^\dagger a_1 + a_2^\dagger a_2)} = \tau_-^\dagger$$

$$\alpha_+ = a_1^\dagger a_2 = \alpha_-^\dagger$$

$$\beta_+ = a_2^\dagger \sqrt{N-(a_1^\dagger a_1 + a_2^\dagger a_2)} = \beta_-^\dagger$$

$$\tau_0 = a_1^\dagger a_1 + \tfrac{1}{2}(a_2^\dagger a_2 - N)$$

$$\Gamma = N/3 - a_2^\dagger a_2$$

as the 8 generators of SU(3).

Define the squeezed state $|\lambda,\mu\rangle = \eta^{-1} \exp(\lambda\tau_+) \exp(\mu\beta_+) |0,0\rangle$ where the normalization η is given by $(1+\lambda^2+\mu^2)^{\frac{1}{2}N}$. We evaluate the uncertainties ΔX, ΔP for the coordinates

$$X = (A + A^{\dagger})/\sqrt{2} \quad , \quad P = (A - A^{\dagger})/i\sqrt{2}$$

where $A = a_1 \cos\vartheta + a_2 \sin\vartheta$. We have, as usual,

$$(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2 = \tfrac{1}{2} + \langle A^{\dagger} A \rangle + \langle A^{\dagger 2} \rangle - 2\langle A^{\dagger} \rangle^2$$

where all expectations are with respect to the state $|\lambda, \mu\rangle$, with λ and μ real. After some computation we obtain

$$\langle A^{\dagger} \rangle = \frac{\lambda \cos\vartheta + \mu \sin\vartheta}{(1+\lambda^2+\mu^2)^N} \sum_{i,j=0}^{N} \mu^{2i} \lambda^{2j} \binom{N}{i\,j} \sqrt{N-i-j}$$

$$\langle A^{\dagger 2} \rangle = \frac{(\lambda \cos\vartheta + \mu \sin\vartheta)^2}{(1+\lambda^2+\mu^2)^N} \sum_{i,j=0}^{N} \mu^{2i} \lambda^{2j} \binom{N}{i\,j} \sqrt{(N-i-j)(N-i-j-1)}$$

$$\langle A^{\dagger} A \rangle = \frac{N}{(1+\lambda^2+\mu^2)} (\lambda \cos\vartheta + \mu \sin\vartheta)^2$$

where $\binom{N}{ij} = N!/[i!\ j!\ (N-i-j)!]$. Without loss of generality we may choose $\lambda = \alpha \cos\varphi$, $\mu = \alpha \sin\varphi$; and note that the summations are independent of φ . Thus

$$(\Delta X)^2 = \frac{1}{2} + \cos^2(\vartheta-\varphi) \left\{ \frac{N\alpha^2}{1+\alpha^2} + \frac{\alpha^2}{(1+\alpha^2)^N} \sum_{\ell=0}^{N} \alpha^{2\ell} \binom{N}{\ell} \sqrt{(N-\ell)(N-\ell-1)} - \right. \tag{3}$$
$$\left. - \frac{2\alpha^2}{(1+\alpha^2)^{2N}} \left[\sum_{\ell=0}^{N} \alpha^{2\ell} \binom{N}{\ell} \sqrt{N-\ell} \right]^2 \right\}$$

where the term in $\{\ \}$ of equation (3) is precisely the value (putting $N=2\sigma$) of $(\Delta X)^2 - \frac{1}{2}$ for the SU(2) one-photon squeezed state[6]. Thus, if squeezing occurs in the SU(2) case, this $\{\ \}$ term must be negative; and so the optimum SU(3) squeezing occurs for $\vartheta = \varphi$, giving the SU(2) result.

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